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Nilpotent invariants in $N = 4$ SYM

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Abstract

It is shown that there are no nilpotent invariants in $N = 4$ analytic superspace for $n \leq 4$ points. It is argued that there is (at least) one such invariant for $n = 5$ points which is not invariant under $U(1)_Y$. The consequences of these results are that the $n = 2$ and 3 point correlation functions of the $N = 4$ gauge-invariant operators which correspond to KK multiplets in AdS supergravity are given exactly by their tree level expressions, the 4 point correlation functions of such operators are invariant under $U(1)_Y$ and correlation functions with $n \geq 4$ points have non-trivial dependence on the Yang-Mills coupling constant.

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In a series of papers it has been suggested that it might be profitable to study Green's functions of certain classes of gauge-invariant operators in four-dimensional superconformal field theories using superspaces which are specially adapted to the superconformal geometry of these classes [1, 2]. The operators of interest are initially given as constrained superfields on Minkowski superspace, but by working in appropriately defined superspaces these constraints can be solved explicitly in a rather simple and geometrically natural fashion [3]. The basic idea is to study the Green's functions of such operators using the superconformal Ward identities considered as differential constraints on multiple products of these superspaces. For the case of $N = 4$ SYM there is a natural class of operators represented by analytic superfields. This class of operators, for the gauge group $SU(N_c)$, coincides with the operators which couple to the Kaluza-Klein multiplets of IIB supergravity on an $AdS_5 \times S^5$ background [4] and is therefore of paramount interest in the context of the Maldacena conjecture relating IIB supergravity on this background to $N = 4$ superconformal field theory [5].

A step towards finding the full consequences of the superconformal Ward identities was taken in reference [2] where superconformal invariants were studied. These invariants generalise the usual cross-ratios which arise as ordinary conformal invariants. However, in a recent paper [6], in which the possible implications of the additional $U(1)_Y$ symmetry of IIB supergravity for $N = 4$ SYM were discussed, Intriligator observed that the non-nilpotent invariants constructed in reference [2] are invariant under this $U(1)_Y$ symmetry. He discovered that if one assumes that the correlation functions themselves are invariant under this symmetry then one is led to the extremely strong conclusion that they are all independent of the SYM coupling g_{YM}. Intriligator's argument may be paraphrased as follows: part of the $N = 4$ Yang-Mills action is not invariant under the $U(1)_Y$ symmetry and hence differentiating the action with respect to the coupling constant results in this non-invariant term among others. As a result, differentiating any correlator, assumed to be $U(1)_Y$ invariant, with respect to the coupling leads to a correlator with the non-invariant term inserted as an additional operator. However, this new correlator is not $U(1)_Y$ invariant and so by assumption vanishes. It then follows that the original correlator does not depend on the coupling constant. Intrilligator then concluded that either a) the strong conclusion could be right, b) the harmonic superspace formalism is flawed, or c) there are more invariants than those listed in [2].

Possibility (a) can be ruled out since although this might be feasible for $n \leq 3$ points it is certainly not true for $n \geq 4$ points as explicit calculations have shown [7, 8]. We shall comment briefly below on option (b), but the main point of this note is to argue that possibility (c) holds. This is bound to be the case since the action contains a $U(1)_Y$ non-invariant piece which will lead to non-invariant vertices in the Feynman rules and so to non-invariant Green's functions. In fact, we show that there is almost certainly a 5-point invariant whose leading term behaves like λ^4, where the odd coordinates of analytic superspace are denoted λ, π. The invariants given in [2] all involve these variables in the combination $\lambda\pi$. In $N = 2$ such a dependence of the invariants on the odd coordinates is dictated by R-symmetry, but in $N = 4$ it is not because the R-symmetry group is $SU(4)$ and not $U(4)$. Nevertheless, the $\mathbb{Z}_4$ centre of $SU(4)$ does place restrictions on the way that the odd variables appear in invariants. If e denotes the generator of $\mathbb{Z}_4$, where $e^4 = 1$, one finds that $\lambda \to e\lambda$ whereas $\pi \to \bar{e}\pi$. As a result the odd variables can only appear in combinations of the form $\lambda^p \pi^q$ where $q = p \bmod 4$ [1].

The existence of a λ^4 invariant for 5 points explicitly invalidates Intriligator's argument concerning the non-dependence of the correlation functions on g_{YM} for $n \geq 4$ points. However, we shall also show that there are no nilpotent invariants for $n \leq 4$ points so that all possible 4 point invariants are included correctly in [1]. Thus we can apply Intriligator's construction to prove the conjectured non-renormalisation theorem for $n = 3$ (and 2) points. The general form of the 3 point functions of analytic operators was given in [1] and discussed in more detail in [10]; it was shown in [9] that the AdS supergravity amplitudes and the appropriately normalised free SYM correlation functions agree and it was conjectured that this might be true for all values of the 't Hooft coupling $g^2_{YM}N_c$ and even for all N_c. Subsequently this was confirmed to first non-trivial order in perturbation theory [11]. Moreover, there is an anomaly argument which establishes the strong form of this conjecture for the correlator of three supercurrent multiplets [10] and an alternative argument based on the Adler-Bardeen theorem [12, 13]. One of the main results of this paper is that the strong conjecture holds for all 3 point functions of analytic operators.

Of course it is still possible the option (b) holds, namely that the analytic harmonic superspace formalism is in some way flawed. One concern is that the underlying SYM multiplet is on-shell, and indeed it is true that the analyticity of the composite operators under consideration depends on the field equations of the underlying Yang-Mills fields being satisfied. A consequence of the on-shell nature of the formalism is that it is almost impossible to check analyticity directly in $N = 4$ perturbation theory. Nevertheless, if one considers the $N = 4$ theory as an $N = 2$ theory consisting of a vector multiplet and a hypermultiplet, both transforming under the adjoint representation of the gauge group, it is possible to carry out perturbative calculations in an off-shell $N = 2$ harmonic superspace formalism and it has been verified that analyticity does indeed hold for correlation functions of hypermultiplet composites in low orders in perturbation theory [10, 7]. The present situation is therefore that analyticity in the $N = 4$ formalism should be regarded as an assumption, but that it is supported by the checks in $N = 2$ perturbation theory that have been carried out so far.

We briefly recall the analytic superspace formalism. $N = 4$ analytic superspace $\mathbb{M}$ has coordinates

$$X = \begin{pmatrix} x^{\alpha\dot{\alpha}} & \lambda^{\alpha a'} \\ \pi^{a\dot{\alpha}} & y^{aa'} \end{pmatrix} \tag{1}$$

where each index can take on 2 values. The even coordinates x and y are coordinates for complex spacetime and the internal space $S(U(2) \times U(2))\backslash SU(4)$ respectively. The odd coordinates λ and π number 8 in all, half the number of odd coordinates of $N = 4$ super Minkowski space. An infinitesimal superconformal transformation takes the form

$$\delta X = \mathcal{V}X = B + AX + XD + XCX \tag{2}$$

where each of the parameter matrices is a $(2|2) \times (2|2)$ supermatrix and where

$$\delta g = \begin{pmatrix} -A & B \\ -C & D \end{pmatrix} \in \mathfrak{sl}(4|4) \tag{3}$$

One can show that the central elements in the superalgebra $\mathfrak{sl}(4|4)$ do not act on $\mathbb{M}$ so that one really has an action of the superalgebra $\mathfrak{psl}(4|4)$.

The gauge-invariant operators are $A_q = \mathrm{tr}(W^q)$ where W is the $N = 4$ SYM field strength tensor which takes its values in the Lie algebra $\mathfrak{su}(N_c)$ of the gauge group. These operators transform as

$$\delta A_q = \mathcal{V}A_q + q\Delta A_q \tag{4}$$

where $\Delta = \mathrm{str(A + XC)}$. A correlation function of such operators

$$G(X_1, \ldots X_n) =< A_{q_1}(X_1) \ldots A_{q_n}(X_n) > \tag{5}$$

should satisfy the Ward identity

$$\sum_{i=1}^{n}(\mathcal{V}_i + q_i\Delta_i)G = 0 \tag{6}$$

Such a correlation function, if it does not vanish at zeroth order in the odd variables, can be written in the form

$$G = \text{prefactor} \times F, \tag{7}$$

where the prefactor is a function of the "propagators"

$$g_{ij} = \text{sdet} X_{ij}^{-1} = \frac{\hat{y}_{ij}^2}{x_{ij}^2}, \tag{8}$$

with $X_{ij} = X_i - X_j$ and $\hat{y}_{ij} = y_{ij} - \pi_{ij} x_{ij}^{-1} \lambda_{ij}$, which absorbs the charges of the operators and which is analytic in the internal bosonic coordinates. The function F is therefore a function of superconformal invariants.

In [2] a large number of superinvariants was found, but, as pointed out in [6], they all depend on the odd variables in the combination $\lambda\pi$. They are thus invariant under $PGL(4|4)$ and not just $PSL(4|4)$. To examine whether this list is complete or not we shall look for nilpotent invariants using the supersymmetry Ward identities in a straightforward manner. Suppose F is a nilpotent invariant, then $F = F_o +$ higher order in λ, π, where F_o is itself nilpotent and has a fixed power of the odd variables. To rule out the existence of any such F for a given number of points it is therefore sufficient to show that all possible leading terms F_o vanish. The superconformal Ward identities must hold order by order in powers of the odd variables λ and π so that the action of the Ward identity on F_0 at lowest order is obtained when the Ward identity operator is linearised appropriately with respect to λ and π. These truncated superconformal Ward identities involve the following simplified (linearised) superconformal Killing vectors:

$$\mathcal{V}_{\alpha\dot{\alpha}} = \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} \tag{9}$$

$$\mathcal{V}(D) = x^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + \frac{1}{2}\lambda^{\alpha a'}\partial_{\alpha a'} + \frac{1}{2}\pi^{a\dot{\alpha}}\partial_{a\dot{\alpha}} \tag{10}$$

$$\mathcal{V}^{\alpha\dot{\alpha}} = x^{\beta\dot{\alpha}}x^{\alpha\dot{\beta}}\partial_{\beta\dot{\beta}} + x^{\beta\dot{\alpha}}\lambda^{\alpha b'}\partial_{\beta b'} + \pi^{b\dot{\alpha}}x^{\alpha\dot{\beta}}\partial_{b\dot{\beta}} \tag{11}$$

corresponding to translations, dilations and conformal boosts as well as three similar equations with x and y interchanged. The linearised Killing vectors for ordinary (Q) supersymmetries and special (S) supersymmetries are given respectively by

$$\mathcal{V}_{\alpha a'} = \frac{\partial}{\partial \lambda^{\alpha a'}} \tag{12}$$

$$\mathcal{V}_{\alpha}^{a} = y^{aa'}\frac{\partial}{\partial \lambda^{\alpha a'}} \tag{13}$$

$$\mathcal{V}_{a'}^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\frac{\partial}{\partial \lambda^{\alpha a'}} \tag{14}$$

$$\mathcal{V}^{a\dot{\alpha}} = x^{\alpha\dot{\alpha}}y^{aa'}\frac{\partial}{\partial \lambda^{\alpha a'}} \tag{15}$$

together with a similar set with λ replaced by π. The above superconformal Killing vectors can be read off from the full superconformal Killing vectors given in [1]. The Ward identity also involves the quantity Δ; however, except for dilations and conformal boosts in x and y this term vanishes when linearised.

Translational symmetries of the form of (??) hold for all of the coordinates, so that we can conclude immediately that an invariant can only depend on the differences $X_{ij} := X_i - X_j$.

In order to solve equation (15) we change variables from the differences λ_{ij} to the variables

$$\lambda_{123} := x_{12}^{-1}\lambda_{12} - x_{23}^{-1}\lambda_{23}, \quad \lambda_{234} := x_{23}^{-1}\lambda_{23} - x_{34}^{-1}\lambda_{34}, \dots \tag{16}$$

and

$$\lambda_c = \sum_i x_{ii+1}^{-1}\lambda_{ii+1} \tag{17}$$

It is straightforward to verify that the variables of the first equation are inert under the transformation induced in equation (15) and so this equation implies that the F_0 part of the Green's functions does not depend on λ_c.

Equation (14) can be solved in a similar manner. We introduce the variables

$$\lambda_{1234} := (x_{12}^{-1}y_{12} - x_{23}^{-1}y_{23})^{-1}\lambda_{123} - (x_{23}^{-1}y_{23} - x_{34}^{-1}y_{34})^{-1}\lambda_{234}, \dots \tag{18}$$

which are inert under the transformation induced by (14) and the final variable which is the sum of terms over all points of terms of this form. Equation (14) then implies that F_0 only depends on $\lambda_{1243} \dots$.

Finally, we turn to equation (15) which is the most complicated supersymmetry transformation. It implies that F_0 depends on only λ_{12345} where λ_{12345} is given by

$$\lambda_{12345} = A_{1234}^{-1}\lambda_{1234} - A_{2345}^{-1}\lambda_{2345}, \dots \tag{19}$$

where A_{1234} is obtained by substituting in the supersymmetry Ward identity of equation (15) to bring it to the form

$$\left(A_{1234}\frac{\partial}{\partial\lambda_{1234}} + A_{2345}\frac{\partial}{\partial\lambda_{2345}}\right)F_o = 0 \tag{20}$$

Explicitly

$$\begin{aligned} A_{1234} \;=\; & \Big(y_{12}z_{123}^{-1}x_{12}^{-1}(x_1+x_2) - y_{23}z_{123}^{-1}x_{23}^{-1}(x_2+x_3) \\ & +(y_1+y_2)z_{123}^{-1}x_{12}^{-1}x_{12} - (y_2+y_3)z_{123}^{-1}x_{23}^{-1}x_{23}\Big) - \Big(\text{same with } (123) \to (234)\Big) \end{aligned} \tag{21}$$

where

$$z_{123} := y_{12}x_{12}^{-1} - y_{23}x_{23}^{-1} \tag{22}$$

Moreover it is possible to invert A_{1234} and thus obtain a rational expression for λ_{12345}. This expression is rather complicated, however, so that we shall not give it here. We find essentially identical results with λ replaced by π if we use the corresponding Ward identities.

The conformal boost Ward identity and a similar identity in the internal space are now automatically satisfied as a result of taking the anti-commutators of the Q and S supersymmetry transformations leaving only the Ward Identities for spacetime and internal dilations. However, these are easily solved and just determine the overall power of x and y respectively.

Let us carry out a count of the spinor variables that a n point invariant or Green's functions can depend on. Initially, an invariant depends on all the spinor coordinates $\lambda_i,\ \pi_i,\ i = 1, \dots n$ that is $4n$ λ's and $4n$ π's. The translational supersymmetries (13) imply that it can only depend on differences, that is, on the $4(n-1)$ λ_{ii+1}'s with a similar result for π's. In a similar way, equations (14), (15) and (15) imply that an invaraint can only really depend on $4(n-4)$ spinors of equation (19) with a similar result for π's. Hence for Green's functions with $n \leq 4$ points there are in effect no available spinors with which to form nilpotent invariants.

We now discuss in more detail these consequences of the superconformal Ward identities for Green's functions with a small number of points. The simplest example is at 2 points. F_o can only depend on X_{12}. Then (**??**) implies in this case that

$$x_{12}^{\alpha\dot{\alpha}} \frac{\partial}{\partial \lambda_{12}^{\alpha a'}} F_o = 0 \tag{23}$$

and this in turn implies that F_o cannot depend on λ_{12}, or, by a similar argument π_{12}. Hence there can be no nilpotent 2 point invariants.

Three-point correlation functions have been studied in [10]. If the sum of the charges, $Q = \sum_i q_i$, is even, the result is

$$< A_{q_1} A_{q_2} A_{q_3} >= C_{q_1 q_2 q_3} (g_{12})^{k_1} (g_{23})^{k_2} (g_{31})^{k_3} \tag{24}$$

where C is a constant and where

$$k_1 = \frac{1}{2}(q_1 + q_2 - q_3) \tag{25}$$

$$k_2 = \frac{1}{2}(q_2 + q_3 - q_1) \tag{26}$$

$$k_3 = \frac{1}{2}(q_3 + q_1 - q_2) \tag{27}$$

This solution is unique up to the constant involved because there are no 3 point invariants. There are clearly no non-nilpotent 3 point invariants because the leading term of such an invariant would be either a 3 point spacetime invariant or a 3 point internal space invariant and there are no such objects. To examine the existence of nilpotent invariants we use the above method. This time (15) implies that F_o can depend on λ_{123} and a similar π variable, but (14) shows that this dependence must be trivial. Thus there are no 3 point invariants. An identical argument can be used to show that the 3 point functions with odd total charge, and which consequently vanish to leading order, must in fact vanish to all orders. Without loss of generality we can take such a 3 point function to be specified by charges $q_1, q_2, q_3 + 1$, where $\sum_i q_i$ is again even. It can be written

$$< A_{q_1} A_{q_2} A_{q_3+1} >= (g_{12})^{k_1} (g_{23})^{k_2} (g_{31})^{k_3} F \tag{28}$$

where F is nilpotent and satisfies

$$(\sum_i (\mathcal{V}_i) + \Delta_3) F = 0 \tag{29}$$

However, since the Δ_3 term vanishes for linear S and Q supersymmetries we can immediately deduce that F_0 and therefore also F vanish.

For 4 points, equations (13) and (15) imply that the leading term, F_o, of a putative nilpotent invariant F depends only on the odd variables of equation (16) and a similar pair of π variables, while (14) implies that F_o should only depend on the λ_{1234} spinor of equation (18) as well as a similarly defined π_{1234}. Implementing finally the non-linear S supersymmetry (15) we find that this dependence is actually trivial, and so we conclude that there can be no nilpotent invariants for 4 points. This means that the 4 point invariants are determined by their leading, purely bosonic terms. These are conformal invariants in x and y, that is, cross-ratios of the x and

y differences. There are two independent such variables in both sectors for 4 points and so 4 independent invariants altogether. They may be expressed in terms of the superinvariants given in [2]; for example, one could take as a basis set two super cross ratios of the form

$$\frac{\mathrm{sdet}\, X_{14}\, \mathrm{sdet}\, X_{23}}{\mathrm{sdet}\, X_{12}\, \mathrm{sdet}\, X_{34}} \tag{30}$$

and 2 supertraces of the form

$$\mathrm{str}\,(X_{12}^{-1} X_{23} X_{34}^{-1} X_{41}) \tag{31}$$

An important conclusion of this analysis is that the invariants listed in [2] are indeed complete for 4 points, so that these invariants are in fact invariant under $PGL(4|4)$ and not just $PSL(4|4)$.

Applying a similar argument to the 5 point case we find that the leading term, F_o, of a 5 point nilpotent invariant can only depend on the odd variables λ_{12345} of equation (19) and a similar π_{12345} variable At first sight this expression seems to involve dependence on x_i and not just the differences x_{ij}, but in fact this turns out not to be the case although λ_{12345} as defined in (19) has complicated x-dependent singularities in the y_{ij} which are not allowed in correlation functions. However, these can be removed by simply multiplying through by the denominator, a procedure which does not affect the supersymmetry analysis given above at this lowest order. Although the Green's functions are not necessarily invariant under $U(1)_Y$ they are invariant under $\mathbb{Z}_4$ and so the leading term of a five-point nilpotent invariant can be of the form λ^4 or of the form π^4. From λ_{12345} of (19) we can construct the leading term of a nilpotent 5 point invariant of the form $\lambda^4_{12345} f(x,y)$, for some appropriate function f of the $x'_{ij}s$ and the $y'_{ij}s$. The dilation Ward identity and a similar identity for the internal variable y imply that the dependence of f on x and y is schematically of the form $x^{-2}y^{-2}$. Although we have no proof at present that this leading term can be extended to a full 5 point superinvariant it seems highly probable that such an extension exists. We note, in particularly, that such an invariant is not $U(1)_Y$ invariant as it does not depend on the odd variables as a power series in $\lambda\pi$.

To obtain the consequences of the above results on nilpotent invariants we briefly review Intriligator's argument concerning the dependence of correlation functions on the coupling. To do this in our formalism we first note that the supercurrent $T = \mathrm{tr}(W^2)$ is also the (on-shell) Lagrangian mulitplet. In (real) Minkowski superspace the supercurrent multiplet is $T_{ij,kl} := \mathrm{tr}(W_{ij}W_{kl})_{20}$, where $i,j = 1,\ldots 4$ now denote $SU(4)$ indices and where the subscript 20 indicates that the real twenty-dimensional representation is to be projected out from the product by imposing the condition $\epsilon^{ijkl}T_{ij,kl} = 0$. Reality means that

$$T^{ij,kl} := \frac{1}{4}\epsilon^{ijmn}\epsilon^{klpq}T_{mn,pq} = \bar{T}^{ij,kl} \tag{32}$$

If we set

$$T'^{ij,kl} := T^{ik,jl} + T^{jk,il} \tag{33}$$

then T' is symmetric on both pairs of indices, symmetric under the interchange of the pairs and vanishes on symmetrisation over any 3 indices. Using T' we can form a complex superaction [14]

$$\mathcal{S} = \int d^4x\, D_{ij} D_{kl} T'^{ij,kl} \tag{34}$$

where

$$D_{ij} := D_{\alpha i} D^{\alpha}_j = D_{ji} \tag{35}$$

The invariance of $\mathcal{S}$ under supersymmetry follows from the constraints

$$D_{\alpha i}T'^{jk,lm} = 2\delta_i^{(j}\Lambda_\alpha^{k),lm} = 2\delta_i^{(l}\Lambda_\alpha^{m),jk} \tag{36}$$

where

$$\Lambda_\alpha^{j,kl} := \frac{1}{4}D_{\alpha i}T'^{ij,kl} \tag{37}$$

and

$$\bar{D}_{\dot\alpha}^{(i}T'^{jk),kl} = 0 \tag{38}$$

These constraints are themselves a consequence of the constraints satisfied be the underlying field strength W_{ij}:

$$\nabla_{\alpha i}W_{jk} = \nabla_{\alpha[i}W_{jk]} \tag{39}$$

where $\nabla_{\alpha i}$ is the gauge-covariant spinorial derivative. It is straightforward to compute that

$$\mathcal{S} = \int d^4x\ (-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \ldots) \tag{40}$$

$F_{\alpha\beta}$ being the self-dual part of the Minkowski space Yang-Mills field strength tensor.

The analytic supercurrent T is related to $T_{ij,kl}$ by means of the $SU(4)$ harmonic variables $(u_r{}^i,\ u_{r'}{}^i) \in SU(4),\ r = 1,2,\ r' = 3,4.$

$$T = \frac{1}{4}\epsilon^{rs}\epsilon^{tu}u_r{}^iu_s{}^ju_t{}^ku_u{}^lT_{ij,kl} \tag{41}$$

We can therefore rewrite the superaction as a harmonic superaction

$$\mathcal{S} = \int d\mu\ T \tag{42}$$

where

$$d\mu := d^4x\, du\, (D')^4 \tag{43}$$

with

$$D' \sim D_{\alpha r'} := u_{r'}{}^iD_{\alpha i} \tag{44}$$

and where du denotes the standard invariant measure on the coset $S(U(2)\times U(2))\backslash SU(4)$. Using this formalism we may then write the on-shell action as

$$S = \mathrm{Im}\left(\tau\int d\mu\, T\right) \tag{45}$$

where τ is the coupling

$$\tau := \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2} \tag{46}$$

Note that, in terms of the odd variables λ, π the measure $d\mu$ essentially contains the factor $d^4\lambda$.

If we differentiate a correlation function with respect to the coupling we get

$$\frac{\partial}{\partial\tau} < \mathcal{A}_{q_1}\ldots\mathcal{A}_{q_n} > \sim \int d\mu\ < \mathcal{T}\mathcal{A}_{q_1}\ldots\mathcal{A}_{q_n} > \tag{47}$$

For example, for a 3 point correlator we have

$$\frac{\partial}{\partial\tau} < \mathcal{A}_{q_2}\mathcal{A}_{q_3}\mathcal{A}_{q_4} > \sim \int d\mu_1\ < \mathcal{T}(1)\mathcal{A}_{q_2}\mathcal{A}_{q_3}\mathcal{A}_{q_4} > \tag{48}$$

where $\mathcal{A}_q = (g_{YM})^{-q} A_q$. The differentiation of the path integral expression for the correlation function sees these explicit factors of g_{YM}, but these terms are cancelled by a term arising from the differentiation of the action which counts the powers of fields in the operators. The residual term from differentiating the action is then the on-shell integrated action. Now the integral on the right projects out the $(\lambda_1)^4$ term in the 4 point correlator. Since this correlator depends on the odd variables only through $\lambda\pi$ the result would have to have the form $\pi^4\times$ a power series in $\lambda\pi$. However, the 3 point correlator also depends only on $\lambda\pi$ and therefore has no such terms. Thus the integral is zero and we conclude that all 3 point correlators have trivial dependence on the coupling.

Such a conclusion will not hold for 4 point correlators, however, provided that the putative 5 point nilpotent invariant discussed above exists. The leading term of this invariant can be expressed in terms of the differences $\lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}$ and so gives a contribution of the integrand of the form $(\lambda_1)^4$ which moreover has no $\pi's$. Therefore one cannot conclude that 4 point functions, or indeed n point functions with $n \geq 4$, should have trivial coupling constant dependence.

To conclude, we have shown that the conjectured non-renormalisation theorem for 2 and 3 point correlation functions in $N = 4$ Yang-Mills theory holds exactly. This lends further support to the Maldacena conjecture to add to the results obtained using instanton techniques [15]. We have also shown that the 4 point invariants are correctly listed in [2], so that the 4 point correlation functions of analytic operators are actually invariant under $U(1)_Y$. We have also seen that it seems likely that there is a 5 point invariant whose leading term behaves like λ^4. Such an invariant would not be invariant under $U(1)_Y$ and its existence would imply that n point functions for $n \geq 4$ do not depend trivially on the coupling.

Note added In a recently posted paper [16] an argument for the 3-point non-renormalisation theorem was given based on a conjecture concerning the behaviour of the OPE. The authors of this paper have suggested, on the basis that it leads to $U(1)_Y$ invariant correlation functions, that the analytic superspace method is flawed. However, as we have shown above, there are now good grounds to suppose that this will not be true for $n \geq 5$ points. These authors also suggest that the analytic superspace formalism is not capable of accommodating the long supermultiplets of the theory. We believe that this is not the case; this point will be discussed further elsewhere.

References

[1] P.S. Howe and P.C. West, *Non-perturbative Green's functions in theories with extended superconformal symmetry*, hep-th/9509140; *OPEs in four-dimensional superconformal field theories*, Phys. Lett. **B389** (1996) 273-279, hep-th/9607060; *Is $N = 4$ Yang-Mills soluble?* (1996) in Moscow 1996 Physics, 622-626, hep-th/9611074.

[2] P.S. Howe and P.C. West, *Superconformal invariants and extended supersymmetry*, Phys. Lett. **B400** (1997) 307-313, hep-th/9611075.

[3] A.Galperin, E.Ivanov, S.Kalitzin, V.Ogievetsky and E.Sokatchev, Class.Quantum Grav. **1** (1984) 469; G.G. Hartwell and P.S. Howe *(N, p, q) harmonic superspace* Int. J. Mod. Phys. **A10** (1995) 3901-3920, hep-th/9412147; *A superspace survey* Class. Quant. Grav. **12** (1995) 1823-1880.

[4] L. Adrianopoli and S. Ferrara, Phys. Lett. **B430** (1998) 248-253, hep-th/9804006.

[5] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, hep-th/9711200; S.S. Gubser, I.R. Klebhanov and A.M. Polyakov, Phys. Lett. B428 (1998) 105; E. Witten, *Anti-de Sitter space and holography*, hep-th/9802150

[6] K. Intriligator, *Bonus symmetries of N=4 superYang-Mills correlation functions via AdS duality*, hep-th/9811047

[7] B. Eden, P.S. Howe, C. Schubert, E. Sokatchev and P.C. West, *Four point functions in N=4 supersymmetric Yang-Mills theory at two loops* hep-th/9811172.

[8] F. Gonzalez-Rey, I. Park, K. Schalm *A Note on four-point functions of conformal operators in $N = 4$ Yang-Mills*, hep-th/9811155

[9] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, *Three-point functions of chiral operators in $D = 4, \mathcal{N} = \triangle$ SYM at large N*, hep-th/9806074.

[10] P.S. Howe, E. Sokatchev and P.C. West, *Three-point functions in N=4 Yang-Mills*, hep-th/9808162.

[11] E. d'Hoker, D.Z. Freedman and W. Skiba, *Field theory tests for correlators in the AdS/CFT correspondence*, hep-th/9807098.

[12] S.S. Gubser and I. Klebanov, *Absorption by branes and Schwinger terms in the worldvolume theory* hep-th/9708005.

[13] D.Z. Freedman, S. Mathur, A. Matusis and L. Rostelli *Correlation functions in the CFT/AdS correpondence*, hep-th/9804058.

[14] P.S. Howe, K.S. Stelle and P.K. Townsend, *Superactions*, Nucl. phys. **B** (1982)

[15] M. Bianchi, M. Green , S. Kovacs and G. Rossi, *Instantons in supersymmetric Yang-Mills and D instantons in IIB superstring theory*, JHEP **08** (1998) 13, hep-th/9807033; N. Dorey, T. Hollowood, V. Khoze, M. Mattis and S. Vandoren, *Mulitinstantons and Maldacena's conjecture* hep-th/9811060.

[16] K. Intriligator and W. Skiba, *Bonus symmetry and the operator product expansion of N=4 Super-Yang-Mills* hep-th/9905020.